Lecture 05

Robot Manipulator Kinematics

Acknowledgement :
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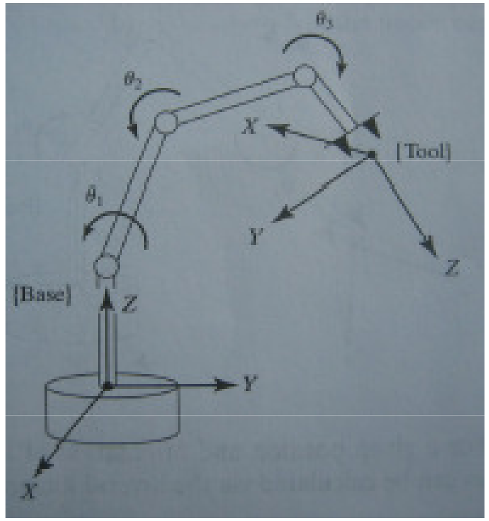
Prof. Harry Asada, AI Laboratory, MIT, USA

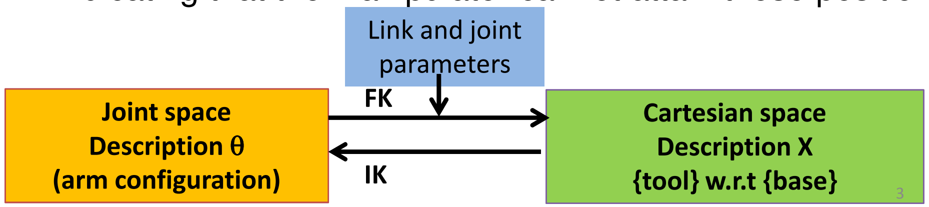
Robot Manipulator Kinematics

- Kinematics is the analysis of motion without regard to the forces/torques that cause the motion.
- Within kinematics, one studies position, velocity, acceleration (and even higher order derivatives of position) w.r.t. time



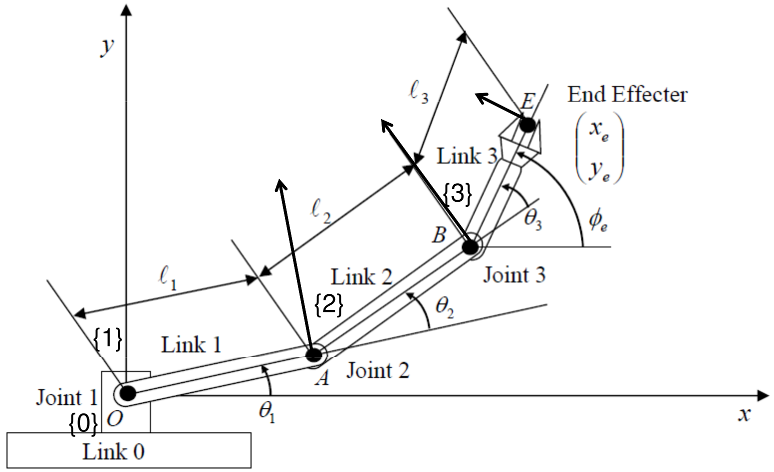
Forward and Inverse Kinematics

- **Forward Kinematics (FK)**
  - Static geometrical problem of computing position and orientation of the end-effector $\mathbf{x} = (x, y, z, \phi_x, \phi_y, \phi_z)^T$ relative to the base frame given the arm configuration $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots \theta_n)^T$
- **Inverse Kinematics (IK)**
  - IK determines the arm configuration $\boldsymbol{\theta}$, for a given position and orientation of the end-effector **x**.
  - IK is the practical problem of manipulator control.
  - IK is often ill-posed, need numerical methods to solve IK.
  - For all **x** outside the workspace, IK produces no solutions $\boldsymbol{\theta}$ indicating that the manipulator cannot attain those position.



Kinematics of Planner Serial Linkages

- Planar kinematics is much more tractable mathematically, compared to general three-dimensional kinematics



Consider the three degree-of-freedom planar robot arm, which consists of one fixed link (link 0) and three movable links that move on the plane. All the links are connected by revolute joints whose joint axes are all perpendicular to the plane of the links.

Kinematics of Planner Serial Linkages

- To describe the robot arm, the following geometric parameters are required
  - Link lengths : l_1, l_2, l_3
- Identify joints and links
  - Actuator 1 couples link0 to link1 and create θ_1
  - Actuator 2 couples link1 to link2 and create θ_2
  - Actuator 3 couples link2 to link3 and create θ_3
- Set up the co-ordinate frame {0} fixed to the base
- Forward Kinematics: describes end-effector position (x_e, y_e) and orientation in terms of joint displacements and link lengths

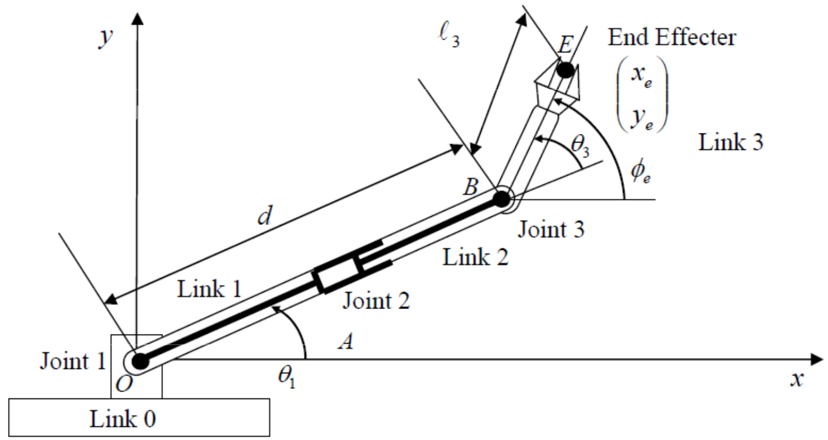
$$x_e = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)$$

$$y_e = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3)$$

$$\phi_e = \theta_1 + \theta_1 + \theta_1$$

Kinematics of Planner Serial Linkages

- Ex: Solve forward kinematics of the following planner RPR manipulator



- Joint displacements: θ_1(rotary), d(prismatic), θ_3(rotary)
- Joint displacement vector: $\boldsymbol{q} = [\theta_1\ d\ \theta_3]^T$

Inverse Kinematics of Planner Manipulators

- Consider the problem of moving the end-effecter of a manipulator arm to a specified position and orientation.
- We need to find the joint displacements that lead the end-effecter to the specified position and orientation.
- To achieve desired end-effector position and orientation, inverse kinematics is solved, and each joint is moved to the determined values through joint controllers. IK is central to **manipulator control problem**.

Inverse Kinematics Problem

- Inverse kinematics is more complex in the sense that multiple solutions may exist for the same end-effecter location
- Since the kinematic equation is comprised of nonlinear simultaneous equations with many trigonometric functions, it is **not always possible** to derive a **closed-form solution.**
- solutions may not always exist for a particular range of end-effecter locations and arm structures
- When the kinematic equation cannot be solved analytically, **numerical methods** are used in order to derive the desired joint co-ordinates.

Inverse Kinematics

$x_w = x_e - l_3 \cos \phi_e$

$y_w = y_e - l_3 \sin \phi_e$

$l_1^2 + l_2^2 - 2 l_1 l_2 \cos \beta = r^2$

where $r^2 = x_w^2 + y_w^2$

$\Rightarrow \beta = \cos^{-1}\left(\dfrac{l_1^2 + l_2^2 - r^2}{2 l_1 l_2}\right)$

$\Rightarrow \theta_2 = \pi - \beta$

Simillary , $r^2 + l_1^2 - 2 r l_1 \cos \gamma = l_2^2$

$\Rightarrow \gamma = \cos^{-1}\left(\dfrac{r^2 + l_1^2 - l_2^2}{2 r l_1}\right)$

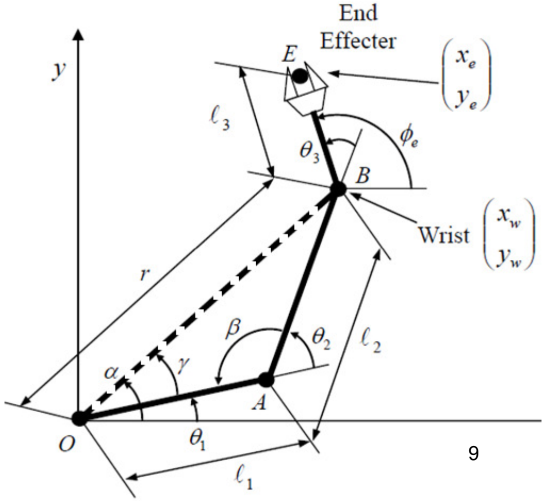
$\alpha = \tan^{-1}\left(\dfrac{y_w}{x_w}\right)$

$\Rightarrow \theta_1 = \alpha - \gamma$

Then,

$\theta_3 = \phi_e - \theta_1 - \theta_2$

- Ex: Solve IK of the planner manipulator (given: $x_e\ y_e\ \phi_e$)
- Use two step method (wrist and end) approach



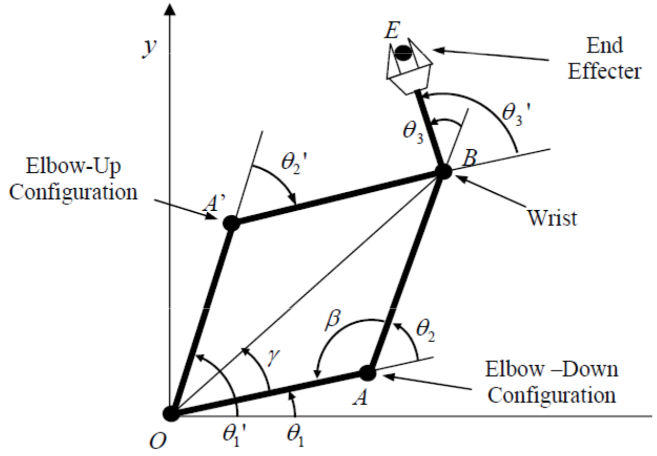
Multiple Solutions

- Interestingly, **elbow up configuration** will also locate the end-effector at the **same position and orientation**. Therefore, there is another IK solution.

$\theta_1' = \theta_1 + 2\gamma$

$\theta_2' = -\theta_2$

$\theta_3' = \theta_3 + 2(\theta_2 - \gamma)$



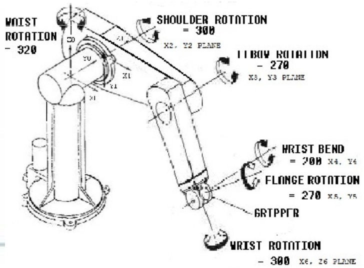
- When there are multiple solutions (arm configurations) for IK, the manipulator is called a **redundant arm**

Multiple Solutions

- The existence of multiple solutions provides the robot with an extra degree of flexibility. Consider a robot working in a crowded environment. If multiple configurations exist for the same end-effecter position/orientation, the robot can choose the most appropriate, collision-free configuration.
- Each IK solution must satisfy joint range limits of rotary joints and stroke limits of prismatic joints.

PUMA 560 arm

Six rotary Joints



Joint Limits

-170^0	$\le \theta_1 \le$	170^0
-225^0	$\le \theta_2 \le$	45^0
-250^0	$\le \theta_3 \le$	75^0
-135^0	$\le \theta_4 \le$	135^0
-100^0	$\le \theta_5 \le$	100^0
-180^0	$\le \theta_6 \le$	180^0

IK Parallel Linkages

$x_e = l_1 \cos \theta_1 + l_2 \cos \theta_2$

$y_e = l_1 \sin \theta_1 + l_2 \sin \theta_2$

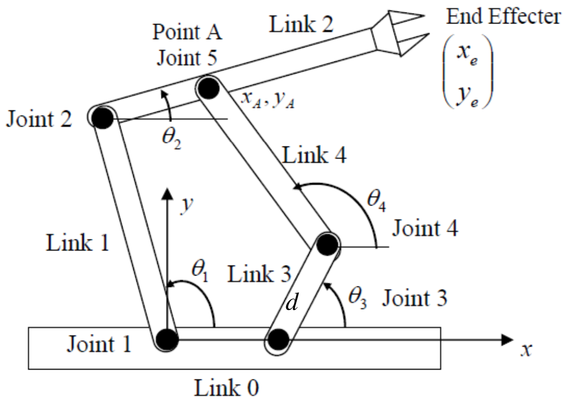
Step 1 (calculate θ_1 and θ_2)

$x_A = l_1 \cos \theta_1 + l_5 \cos \theta_2 = d + l_3 \cos \theta_3 + l_4 \cos \theta_4$

$y_A = l_1 \sin \theta_1 + l_5 \sin \theta_2 = l_3 \sin \theta_3 + l_4 \sin \theta_4$

Step 2 (calculate x_A , y_A)

Step 3 (calculate θ_3 and θ_4)

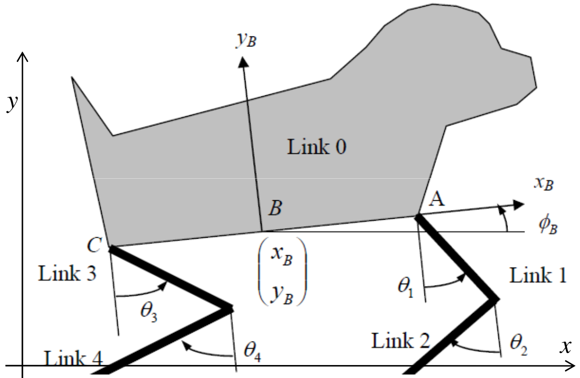


Inverse Kinematics of Parallel Linkages

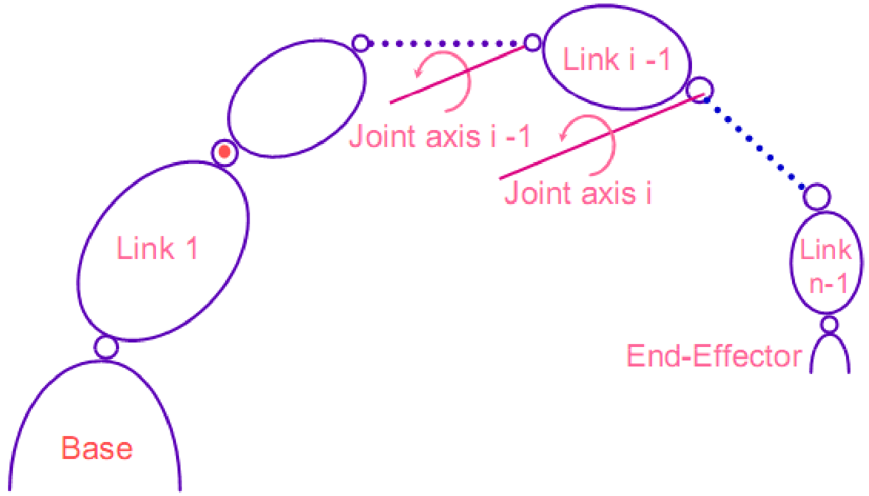
Homework

Solve IK of the doggy robot

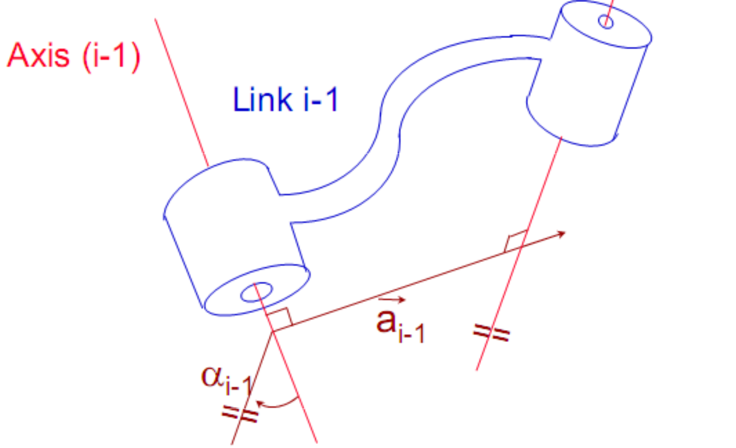
1. Given $(x_B, y_B\ \phi_B) \rightarrow (x_A, y_A)$ and (x_C, y_C)
2. Knowing $(x_A, y_A) \rightarrow$ find θ_1, θ_2
3. Knowing $(x_C, y_C) \rightarrow \theta_3, \theta_4$



Serial Link Manipulators



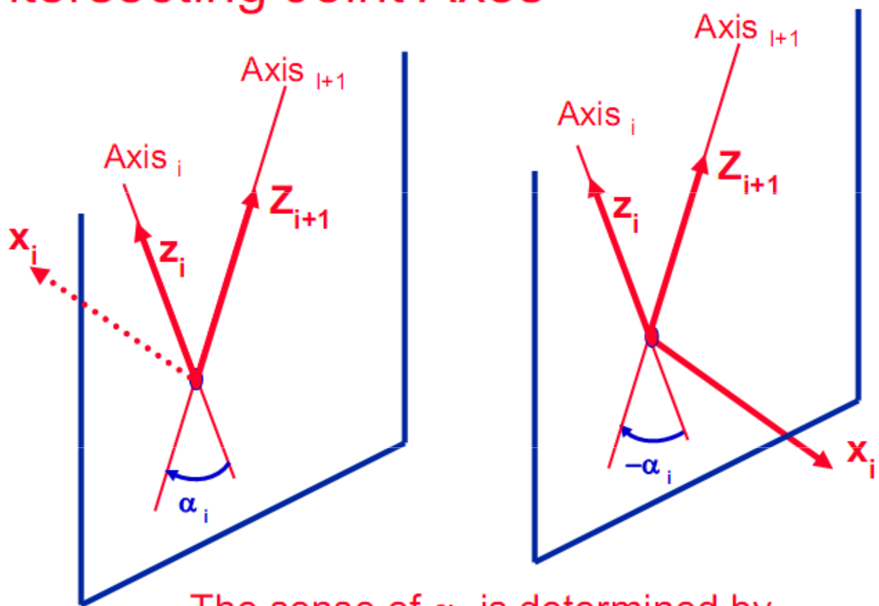
Link Description



a_{i-1}: Link Length - mutual perpendicular
unique except for parallel axis

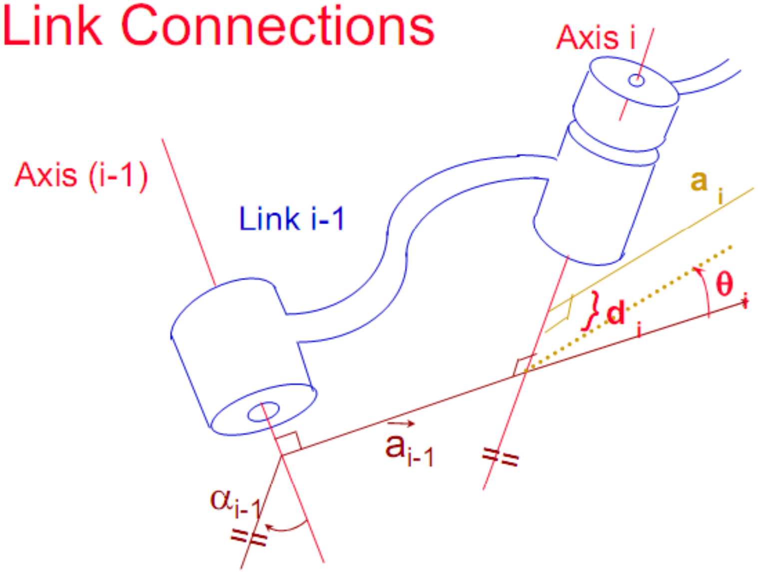
α_{i-1}: Link Twist - measured in the right-hand sense about $\vec{a}_{i-1}$

Intersecting Joint Axes



The sense of α_i is determined by the direction of x

Link Connections

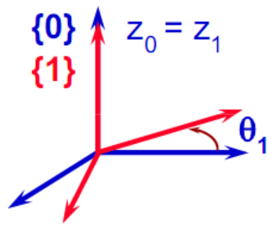


d_i: Link Offset -- variable if joint i is *prismatic*

θ_i: Joint Angle -- variable if joint i is *revolute*

First Link

Revolute



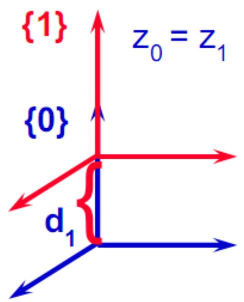
$a_0 = 0$

$\alpha_0 = 0$

$d_1 = 0$

$\theta_1 = 0 \longrightarrow \{0\} \equiv \{1\}$

Prismatic



$a_0 = 0$

$\alpha_0 = 0$

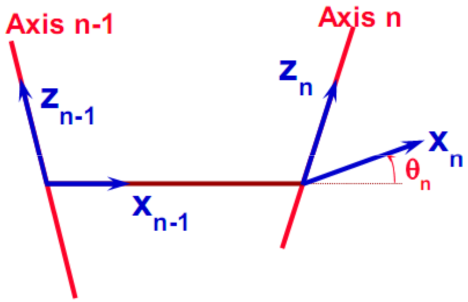
$\theta_1 = 0$

$\mathbf{d_1 = 0} \longrightarrow \{0\} \equiv \{1\}$

Last Link

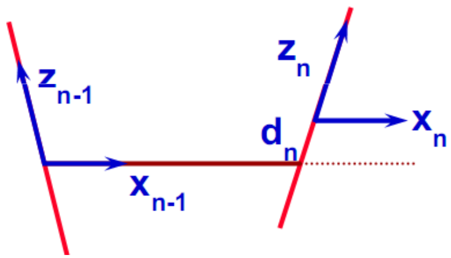
Revolute $d_n = 0$

$\theta_n = 0 \longrightarrow x_n = x_{n-1}$



Prismatic $\theta_n = 0$

$d_n = 0 \longrightarrow x_n = x_{n-1}$



Denavit-Hartenberg Parameters

4 D-H parameters $(\alpha_i, \; a_i, \; d_i, \; \theta_i)$

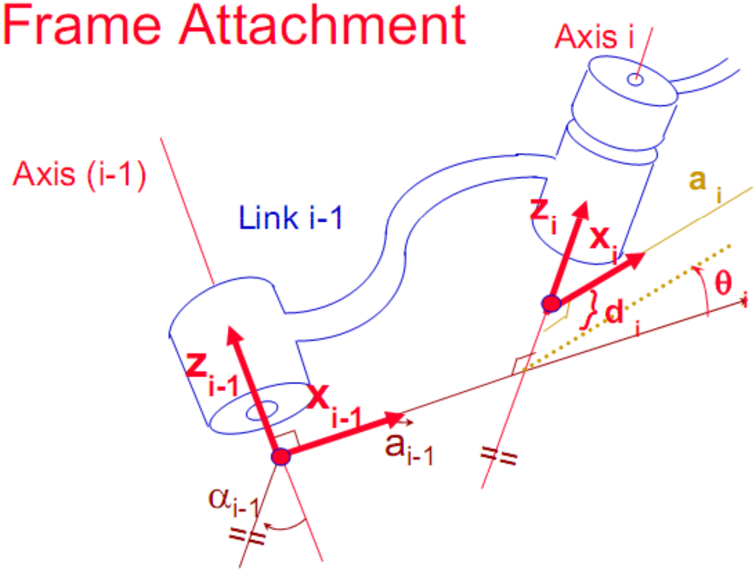
3 fixed link parameters

1 joint variable
- θ_i revolute joint
- d_i prismatic joint

α_i and a_i : describe the Link i

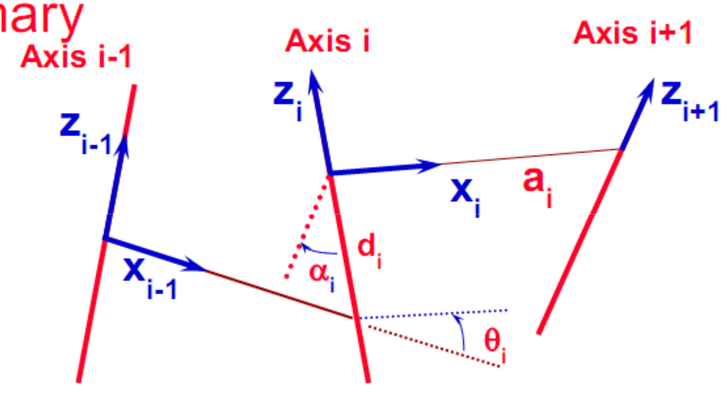
d_i and θ_i : describe the Link's connection

Frame Attachment



y-vectors: complete right-hand frames

Summary



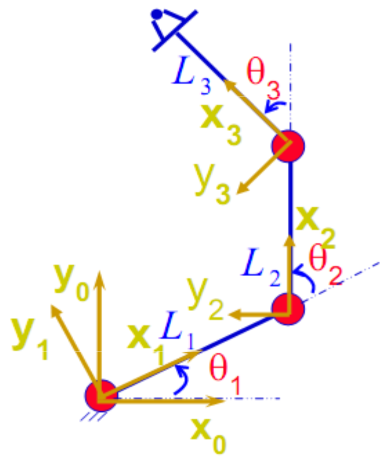
a_i : distance (z_i , z_{i+1}) along x_i

α_i : angle (z_i , z_{i+1}) about x_i

d_i : distance (x_{i-1} , x_i) along z_i

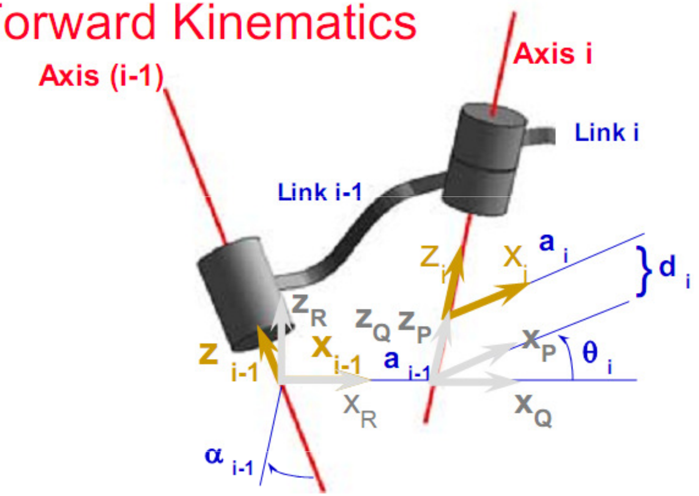
θ_i : angle (x_{i-1} , x_i) about z_i

Class Test: Determine the DH Table of the RRR Planner Robot



i	α_{i-1}	a_{i-1}	d_i	θ_i
1				
2				
3				

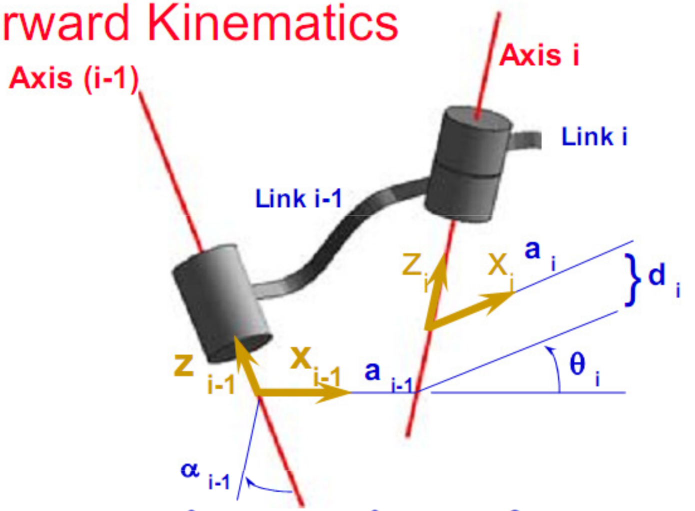
Forward Kinematics



$${}^{i-1}_{i}T = {}^{i-1}_{R}T \; {}^{R}_{Q}T \; {}^{Q}_{P}T \; {}^{P}_{i}T$$

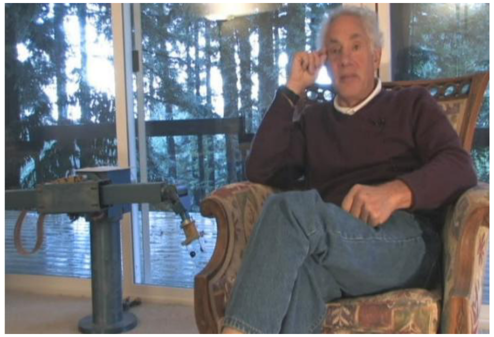
$${}^{i-1}_{i}T_{(\alpha_{i-1},\, a_{i-1},\, \theta_i,\, d_i)} = R_x(\alpha_{i-1})\, D_x(a_{i-1})\, R_z(\theta_i)\, D_z(d_i)$$

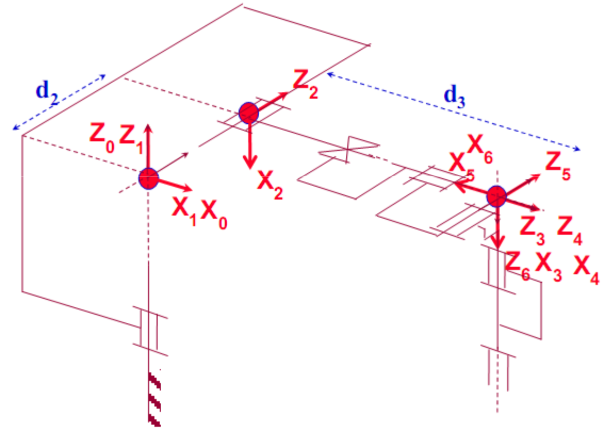
Forward Kinematics



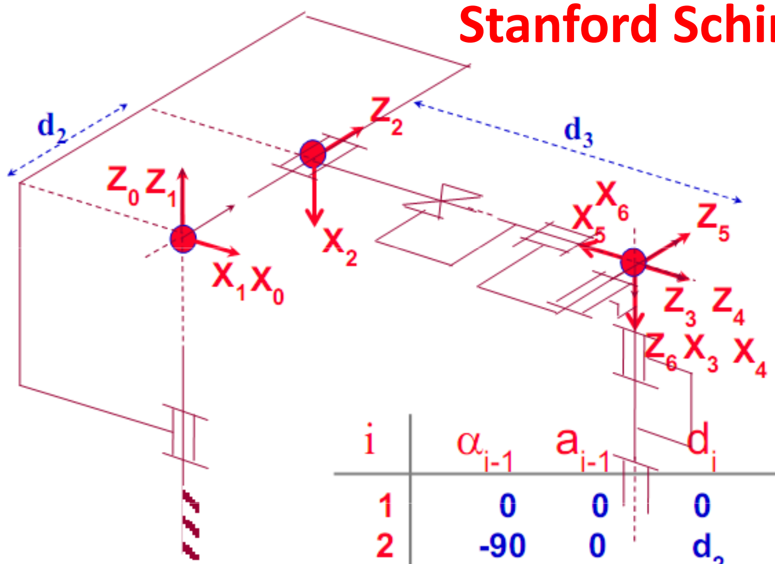
$$ {}^{i-1}_{\ \ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Stanford Scheinman Arm





Stanford Schinman Arm



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

a_i : distance (z_i , z_{i+1}) along x_i
α_i : angle (z_i , z_{i+1}) about x_i
d_i : distance (x_{i-1} , x_i) along z_i
θ_i : angle (x_{i-1} , x_i) about z_i

Stanford Scheinman Arm

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

$$ \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\ s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ -s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Homogeneous transformation between consecuitive joints

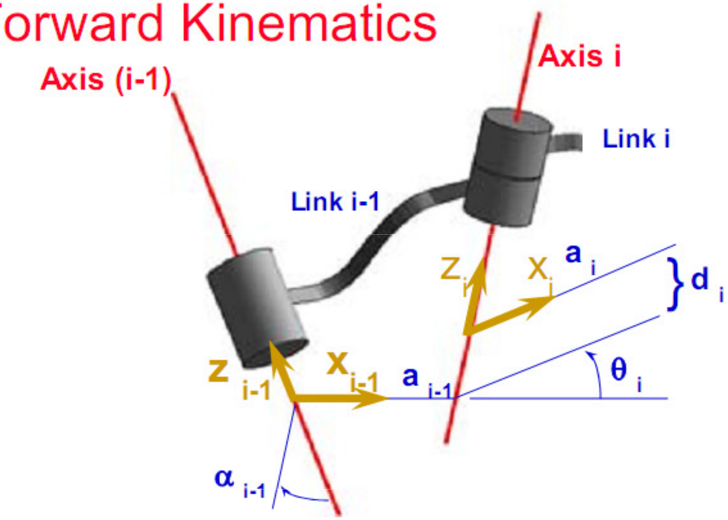
Homogeneous transformation between consecuitive joints

$$^3_4T = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^4_5T = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^5_6T = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward Kinematics



Forward Kinematics: $^0_NT = {}^0_1T \; {}^1_2T \; \dots \; {}^{N-1}_{N}T$

Homogeneous transformation between any two joints

Homogeneous transformation from base to other joints

$$^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0_2T = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 & -s_1d_2 \\ s_1c_2 & -s_1s_2 & c_1 & c_1d_2 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

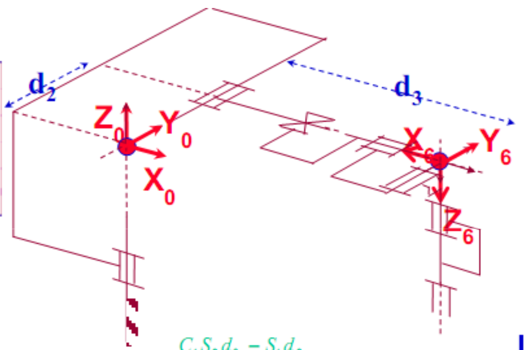
$$^0_3T = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1d_3s_2 - s_1d_2 \\ s_1c_2 & c_1 & s_1s_2 & s_1d_3s_2 + c_1d_2 \\ -s_2 & 0 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0_4T = \begin{bmatrix} c_1c_2c_4 - s_1s_4 & -c_1c_2s_4 - s_1c_4 & c_1s_2 & c_1d_3s_2 - s_1d_2 \\ s_1c_2c_4 + c_1s_4 & -s_1c_2s_4 + c_1c_4 & s_1s_2 & s_1d_3s_2 + c_1d_2 \\ -s_2c_4 & s_2s_4 & c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0_5T = \begin{bmatrix} X & X & -c_1c_2s_4 - s_1c_4 & c_1d_3s_2 - s_1d_2 \\ X & X & -s_1c_2s_4 + c_1c_4 & s_1d_3s_2 + c_1d_2 \\ X & X & s_2s_4 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

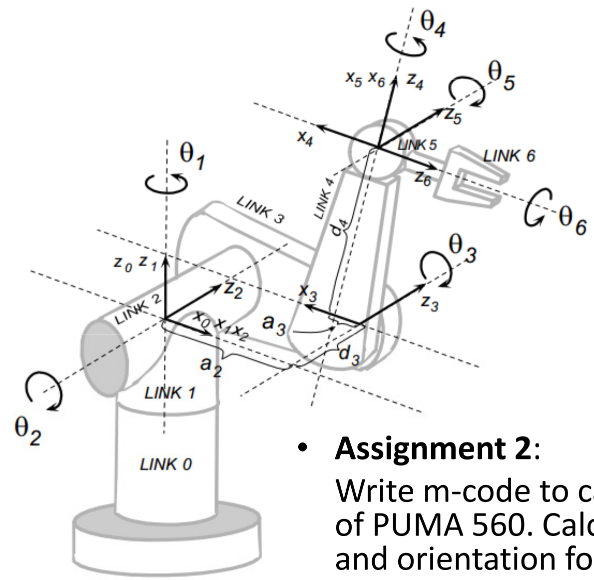
$$^0_6T = \begin{bmatrix} X & X & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2s_5 & c_1d_3s_2 - s_1d_2 \\ X & X & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 & s_1d_3s_2 + c_1d_2 \\ X & X & -s_2c_4s_5 + c_5c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
{}^0_6T = \begin{bmatrix} X & X & c_1c_2c_4s_5 - s_1s_4s_5 + c_1s_2s_5 & c_1d_3s_2 - s_1d_2 \\ X & X & s_1c_2c_4s_5 + c_1s_4s_5 + s_1s_2c_5 & s_1d_3s_2 + c_1d_2 \\ X & X & -s_2c_4s_5 + c_5c_2 & d_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



$$
x = \begin{pmatrix} x_p \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{vmatrix} C_1S_2d_3 - S_1d_2 \\ S_1S_2d_3 + C_1d_2 \\ C_2d_3 \\ C_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] - S_1(S_4C_5C_6 + C_4S_6) \\ S_1[C_2(C_4C_5C_6 - S_4S_6) - S_2S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \\ -S_2(C_4C_5C_6 - S_4S_6) - C_2S_5C_6 \\ C_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6) \\ S_1[-C_2(C_4C_5S_6 + S_4C_6) + S_2S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6) \\ S_2(C_4C_5S_6 + S_4C_6) + C_2S_5S_6 \\ C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5 \\ S_1(C_2C_4S_5 + S_2C_5) + C_1S_4S_5 \\ -S_2C_4S_5 + C_2C_5 \end{vmatrix}
$$

Programmable Universal Manipulator Arm (PUMA) 560



i	α_{i-1}	a_{i-1}	θ_i	d_i	σ_i
1	0	0	θ_1	0	0
2	-90^0	0	θ_2	0	0
3	0	a_2	θ_3	d_3	0
4	-90^0	a_3	θ_4	d_4	0
5	90^0	0	θ_5	0	0
6	-90^0	0	θ_6	0	0

$a_2 = 0.43180\ m$

$a_3 = 0.02032\ m$

$d_3 = 0.12446\ m$

$d_4 = 0.43180\ m$

- **Assignment 2**:
  Write m-code to calculate forward kinematics of PUMA 560. Calculate end-effector position and orientation for a test arm configuration.